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PARTITIONING INTO STEINER SYSTEMS

An *overlarge set of Steiner systems* $S(k-1,k,n)$ is a partition of the set of all k-subsets of an $(n+1)$-set into such systems (each omitting one point). Breach and Street showed that there are just two such sets up to isomorphism for $k = 4$, $n = 8$ both admitting 2-transitive groups. Our purpose here is twofold:

(i) to give a short proof of this result, using the geometry of the $O^+(8,2)$ quadric (including triality);
(ii) to show the non-existence of overlarge sets of $S(5,6,12)$s.

1 Preliminaries

Let $P(k,n)$ denote an overlarge set of $S(k-1,k,n)$ systems, that is, a partition of the set of k-subsets of an $(n+1)$-set into Steiner systems $S(k-1,k,n)$.

Proposition (1) –
(i) $P(k,n)$ consists of $n+1$ Steiner systems, and each point is omitted by one of them.
(ii) If $P(k,n)$ exists, then $P(k-1,n-1)$ exists.
(iii) If $S(k,k+1,n+1)$ exists, then $P(k,n)$ exists.

Proof — The number of Steiner systems is

$$\binom{n+1}{k} / |S(k-1,k,n)| = n+1 .$$

We form the derived system of a $P(k,n)$ with respect to a point p, by taking the derived systems with respect to p of all the Steiner systems involving p; it is easily checked that this is a $P(k-1,n-1)$. Hence by the first sentence (with $n-1$ replacing n), p lies in n of the Steiner systems, and is omitted by just one. Finally if $S(k,k+1,n+1)$ exists, then its $n+1$ derived systems form a $P(k,n)$. □

In this paper, we are concerned only with the cases

$$(k,n) = (4,8) \quad \text{and} \quad (6,12)\,.$$

In both cases, $S(k-1,k,n)$ exists (and is unique) but $S(k,k+1,n+1)$ does not, so the existence question for $P(k,n)$ is non-trivial.

2 The case (k,n)=(4,8)

There is a unique $S(3,4,8)$, whose blocks are the planes of the affine geometry $AG(3,2)$. In this section, we prove the theorem of Breach and Street [1]:

THEOREM (2) – *There are, up to isomorphism, just two $P(4,8)$ systems. Their automorphism groups are the 3-transitive group $P\Gamma L(2,8)$ and the 2-transitive group $ASL(2,3) = V_9SL(2,3)$.*

2.1 The examples

We give a general construction for $P(4,2^d)$ and another construction of a specific $P(4,8)$.

Let $G = P\Gamma L(2,2^d)$ act on the projective line X over $GF(2^d)$. For each $x \in X$,

$$G_x = A\Gamma L(1,2^d) \leq AGL(d,2)\,,$$

the latter group being the automorphism group of the $S(3,4,2^d)$ whose blocks are the planes of the affine space $AG(d,2)$. Let $B(x)$ be the set of blocks of this design. All we have to show is that

$$B(x) \cap B(y) = \emptyset \quad \text{for} \quad x \neq y;$$

in other words, a 4-set determines the x for which it is a block of $B(x)$. But a 4-set which is a block is fixed by a unique V_4 in $PGL(2,2^d)$; and there cannot be two different groups inducing V_4 on it, since the pointwise stabiliser of even three points in $PGL(2,2^d)$ is trivial. Now x is determined as the unique fixed point of this V_4.

The resulting partition admits $P\Gamma L(2,2^d)$ and, since this is a maximal subgroup of the alternating group, it is the full automorphism group of the system.

For the second construction, let V be a 2-dimensional vector space over $GF(3)$, and b a symplectic form on V. (There are just two such forms, one the negative of the other; and the subgroup of $GL(2,3)$ preserving one is

$$Sp(2,3) = SL(2,3)\,.)$$

It is straightforward to check that the following 14 subsets of $V \setminus \{0\}$ form a $S(3,4,8)$:

(i) $\{\pm v, \pm w\}$, where v and w are independent;
(ii) $\{v, w, w+v, w-v\}$, where $b(v,w) = 1$.

If B is this design, the 9 images of B under translations are pairwise disjoint. (We must show that 0 is determined by each set using the affine symplectic structure. For type (i), 0 is the only point of intersection of two diagonals of the quadrangle; the other two pairs of diagonals are parallel. A set of type (ii) consists of a line $L = \{w, w+v, w-v\}$ and a point v off L; the line through v parallel to L is $\{0, v, -v\}$ and the symplectic form distinguishes 0 and $-v$).

This system admits the group $ASL(2,3)$ generated by translations and $SL(2,3)$; as before, this is the full automorphism group. It follows that the two $P(4,8)s$ are not isomorphic.

2.2 The geometry of $0^+(8,2)$

Let V be the 8-dimensionasl vector space consisting of all 9-tuples of even weight. (We identify a vector in V with the subset of $\{1,\ldots,9\}$ which is its support. The weight $wt(v)$, of a vector v is the number of non-zero entries in v, that is, the cardinality of the corresponding set). The usual dot product (mod 2) is a non-degenerate form b on V, and is symmetric. The function

$$f\colon V \longrightarrow GF(2)$$

defined by

$$f(v) = \frac{1}{2}wt(v) \pmod 2$$

is a quadratic form associated with the bilinear form b, since

$$wt(v+w) = wt(v) + wt(w) - 2v \cdot w\,.$$

The points of the quadric Q defined by f are represented by the non-zero singular vectors, i.e. the vectors of weight 4 or 8. A totally singular subspace is a doubly-even self-orthogonal code; the largest dimension of such a subspace is 4, with equality if and only if it is an extended Hamming code, consisting of zero, a vector v of weight 8, and 14 vectors of weight 4 whose supports form a $S(3,4,8)$ on the support of v. The points represented by non-zero vectors in a totally singular 4-dimensional subspace form a *solid* or projective 3-space in Q. This shows that Q is of type $0^+(8,2)$. The solids on Q fall into two families, two solids lying in the same family if and only if their intersection has even codimension.

Consider the set Ω of the 9 vectors of weight 8. These are singular and pairwise non-perpendicular. Since there are 270 solids and each point lies in 30, Ω meets every solid in one point, that is, Ω is an *ovoid*. Moreover, all ovoids are equivalent under the orthogonal group. (For the matrix of inner products of the points of an ovoid is $J-I$ with rank 8, so any 8 of its points form a basis, and any two such bases are equivalent; and the ninth vector is the sum of the other eight.)

The group induced on Ω by its stabiliser is S_9; and

$$S_9 \cap P\Omega^+(8,2) = A_9 .$$

(Here, $PQ + (8,2)$ is the group fixing the two families of solids)

There is a *triality* map (see [2])

$$T : \{\text{points}\} \to \{\text{first family of solids}\}$$
$$\to \{\text{second family of solids}\} \to \{\text{points}\}$$

which maps lines to lines and preserves incidence. If p and q are points, then $p\tau \cap q\tau = \emptyset$ if and only if p and q are not perpendicular. So $Q\tau$ is a set of 9 pairwise disjoint solids, that is, a *spread* of solids. Every spread arises as the image of an ovoid under τ or τ^2. Thus the stabiliser of a spread is A_9.

2.3 Proof of the theorem

As above, we identify a subset X of $\{1,\ldots,9\}$ with the vector in $GF(2)^9$ with support X. Given a $P(4,8)$ system, we know (Proposition 1(i)) that each 8-set supports one of the $S(3,4,8)s$. Adjoining the 8-set to the 14 4-sets of the corresponding $S(3,4,8)$ gives a solid on Q; these 9 solids form a spread. Conversely every spread yields a $P(4,8)$, by simply omitting the vector of weight 8 from each solid. There are

$$|PO^+(8,2) : A_9| = 1920$$

spreads, and hence 1920 $P(4,8)$s. But there are

$$|S_0 : P\Gamma L(2,8)| = 240$$

systems of the first type, and

$$|S_0 : ASL(2,3)| = 1680$$

of the second; so we have them all.

3 The case (k,n)=(6,12)

There is a unique Steiner system $S(5,6,12)$; so it is natural to look for $P(6,12)$s. We show:

THEOREM (3) – *There is no $P(6,12)$ system.*

A feature of the proof is worth noting. We reduce the question to a very specific one about a configuration of subgroups of M_{12}. The non-existence could presumably be shown computationally; we have not done this. But the same reduction shows that, if the partition exists, then it is unique in such a strong sense that we can infer properties (such as transitivity and order) of its automorphism group. But no group of degree 13 can have the properties we find.

3.1 Reduction to 12 points

In the sequel, we use many properties of the unique $S(5,6,12)$ and of its automorphism group M_{12}.

Suppose that a $P(6,12)$ system exists. Let $\{0,1,\ldots,12\}$ be the point set, and let S_i be the Steiner system in the partition which omits the point i. The image T_i of S_i under the transposition $(0\ i)$ is disjoint from S_0 for $i > 0$. (Blocks of T_i not containing i lie also in S_i and hence not in S_0. Suppose that B is a block of T_i containing i. Since the complement of a block of $S(5,6,12)$ is a block, $\{1,\ldots,12\} \setminus B$ is a block of T_i, hence of S_i, hence not of S_0; so B is not a block of S_0 either.) Conversely if T is a system on $\{1,\ldots,12\}$ disjoint from S_0, then its image under $(0\ i)$ is also disjoint from S_0.

The systems T_i are not necessarily disjoint from one another. But it is true that no 6-set disjoint from $\{i,j\}$ can be a block of both T_i and T_j, since such a block would be fixed by both $(0\ i)$ and $(0\ j)$ and so lie in both S_i and S_j. Taking complements, no 6-set containing both points is a block of both systems. In other words,

(i) every block of $T_i \cap T_j$ separates i and j (that is, contains just one of them).

Moreover,

(ii) there is no 5-set C such that $C \cup \{i\} \in T_i$ and $C \cup \{j\} \in T_j$.

Conversely, it is easy to check that if $T_1, \ldots, T_{12}$ are $S(5,6,12)$s on $\{1, \ldots, 12\}$ satisfying (i) and (ii), then $S_i = T_i^{(0\,i)}$ ($i = 1, \ldots, 12$) are disjoint. Since we are proving nonexistence, it suffices to consider (i); in fact, (i) implies (ii).

Summarising, we must show that it is impossible to find systems $T_1, \ldots, T_{12}$, in $1-1$ correspondence with $1, \ldots, 12$ and all disjoint from S, such that, for $i \neq j$, each block of $T_i \cap T_j$ separates i and j. (Here "system" is used to mean the set of blocks of a $S(5,6,12)$ on $\{1, \ldots, 12\}$. For brevity, we shall simply say that $T_i \cap T_j$ separates i and j).

3.2 Subdegrees and intersections

The next task is to describe the systems disjoint from S_0. Since all systems are isomorphic, S_{12} permutes them transitively; the stabiliser of S_0 is its automorphism group M_{12}. Since

$$|S_{12} : M_{12}| = 5040\,,$$

there are 5040 systems, of which half are images of S_0 under even permutations. The systems correspond to cosets of M_{12} in S_{12}, and the suborbits (M_{12}-orbits) to double cosets. Two systems in the same suborbit have isomorphic intersections with S_0. The list is as follows:

	subdegree	$\|S \cap S_0\|$
within A_{12}:	1	132
	440	24
	495	36
	1584	12
outside A_{12}:	66	60
	990	12
	1320	24
	144	0

An outline of the construction of this table follows. Seven of the eight double cosets contain permutations with many fixed points. For example, there are 66 transpositions. All lie in different cosets, since M_{12} contains no element which is the product of just two transpositions. These 66 cosets constitute a suborbit. A block of S_0 lies in $S_0^{(i\,j)}$ if and only if it is fixed setwise by $(i\ j)$, that is, contains both or neither of i and j; and there are 60 such blocks. The first seven suborbits arise thus, from permutations with cycle types $1, 3, 2^2, 5, 2, 4, 2.3$ respectively. (Sometimes a coset contains more than one of these). Counting incidences of 6-sets with systems (there are 5040.132 such) shows that all non-empty intersections are accounted for, so the remaining 144 systems are disjoint from S_0.

Now M_{12} contains a maximal subgroup $PSL(2,11)$ of index 144 whose normaliser in S_{12} in $PGL(2,11)$. Thus, if

$$t \in PGL(2,11)\backslash PSL(2,11)\,,$$

then the coset of M_{12} lies in a suborbit of length

$$|M_{12} : M_{12} \cap M_{12}^t| = 144\,,$$

which must correspond to the systems disjoint from S_0.

Thus the suborbit Σ of length 144 must contain the systems $T_1, \ldots, T_{12}$ we seek.

Any two systems in Σ differ by an even permutation. It is easy to see that, if two systems differ by a permutation of type 3 or 2^2, then their intersection separates no pairs of points. (For example, if (a b c) is a 3-cycle and i and j any two points, then at least one block contains all of a,b,c,i,j). Inspection of an explicit representation of $S(5,6,12)$ shows that, in the case of a 5-cycle f for any point i there is a unique j such that $S \cap S^f$ separates i and j. We conclude that $\{T_1, \ldots, T_{12}\}$ is a 12-clique in the orbital graph corresponding to the suborbit of length 1584.

It is also possible to show that S and S^f satisfy condition (ii) described earlier, proving our claim that (i) implies (ii) (though, as remarked there, we don't need this fact).

3.3 An action of M_{12}

We consider the set Σ of systems disjoint from S_0 and the action of M_{12} on them. This is a familiar permutation representation; we assume facts about the subdegrees and intersection numbers.

As noted, the stabiliser of a point is $PSL(2,11)$ which is maximal in M_{12}. There is another conjugacy class of subgroups isomorphic to $PSL(2,11)$; these are intersections of subgroups M_{11} from the two conjugacy classes of these. A non-maximal $PSL(2,11)$ (G, say) has orbits of length 12 and 132

The subdegrees of M_{12} on Σ are $1, 11, 11, 55, 66$. The orbital graph of valency 66 (the 66-graph) is strongly regular, and E is a clique in this graph. The suborbits of length 11 are paired with each other; so their orbital graphs are directed.

Each of these orbitals is contained in an orbital for S_{12} (indeed, for A_{12}) as described in the last section. We calculate these inclusions. M_{12} acts transitively on the 792 6-sets which are not blocks of S_0, so each set lies in $144.132/792 = 24$ members of Σ. Now for $S, S' \in \Sigma$ with $S \neq S'$, $|S \cap S'|$ depends only on the suborbit containing (S, S'), and is the same for paired suborbits. Summing over all such pairs gives

$$144(22a + 55b + 66c) = 792.24.23\,.$$

Furthermore, the number of edges, in the S_{12}-orbital graph of valency 144, from a point of Σ to points in the suborbits of lengths 440 and 495 are clearly multiples of 5; so $a = c = 12$. Hence $b = 36$. This information determines the inclusions of M_{12}-orbitals on Σ in A_{12}-orbitals. In particular, the orbital with subdegree 55 is not contained in that with subdegree 1584. It follows that our required set of 12 systems must be a coclique in the 55-graph.

We show next that a 12-set which is a coclique in the 55-graph must be a clique in the 66-graph. Let D be such a set, and let x_i be the number of points outside D which are joined to i points of D in the 55-graph. We need two intersection numbers: the number of points z joined to x and y in the 55-graph is 15 or 20 according as x and y are joined in the 11-graph or the 66-graph. Thus, we have

$$\sum x_i = 132\,,$$
$$\sum i x_i = 12.55 = 660\,,$$
$$\sum i(i-1)x_i \leq 12.11.20 = 2640\,,$$

with equality if and only if D is a clique in the 66-graph. But these equations imply

$$\sum (i-5)^2 x_i \leq 0\,,$$

so equality holds (and $x_i = 0$ for $i \neq 5$).

We need the further fact that any 12-clique in the 66-graph is equivalent, under the action of M_{12}, to E. The proof is fairly delicate, and is given in the appendix.

So we may assume that the set $\{T_1, \ldots, T_{12}\}$ is E.

3.4 Completion of the proof

We now have to determine the bijection between E and the point set $\{1, \ldots, 12\}$. The group G of the last section acts transitively on E and fixes a point (say 1); so we are still free to choose $T_1 \in E$ to correspond to 1. Now everything is forced. For, given $S \in E \setminus \{T_1\}$, there is a unique point $i \neq 1$ such that $T_1 \cap S$ separates 1 and i (see 3.2), and by (*) we must have $S = T_i$.

Now the existence question is reduced to the following, completely specific, question: Is it true that (in the above notation) $T_i \cap T_j$ separates i and j, for $i, j > 1$? Indeed, since the set-up is still invariant under the stabiliser of T_1 in G, a Frobenius group of order 55 which acts 2-homogeneously on $\{2, \ldots, 12\}$, it suffices to consider a single pair!

But we can see with no further computation that the system cannot exist. For, if it does exist, we have shown its uniqueness, even up to the choice of the initial point 0; so its automorphism group is transitive. Also, the stabiliser of 0 in the automorphism group is the Frobenius group of order 55 referred to above. But there is no transitive group of degree 13 in which the stabiliser of a point is a Frobenius group of order 55. (For example, such a group would be imprimitive, with blocks of size 2; but 2 doesn't divide 13).

Appendix Cliques in a certain graph

Let Γ be the orbital graph of valency 66 associated with M_{12} acting on the cosets of a maximal $PSL(2,11)$. We show that any 12-clique in Γ is an orbit of a non-maximal $PSL(2,11)$, and hence any two such are equivalent under M_{12}.

We know that there are 144 "good" 12-cliques which are $PSL(2,11)$-orbits; every point lies in 12 of these, and every edge in 2. Let p be a vertex, P its neighbour set, and number the good cliques containing p from 1 to 12. Then points of P can be indexed by 2-subsets of $\{1, \ldots, 12\}$ (q is indexed by the numbers of the two cliques containing p and q); vertices whose labels intersect are adjacent, so the edges of the triangular graph $T(12)$ are contained in the induced subgraph on P. We call the remaining edges within P "strange".

The 12 good cliques through p are permuted by $PSL(2,11)$ in its usual permutation representation. The stabiliser of $\{1,2\}$ is dihedral of order 10, and acts regularly on the other 10 good cliques. So we can identify the 10 points of P joined to $\{1,2\}$ by strange edges with a Cayley graph for D_{10}, of valency 2, and so either a 10-cycle or two 5-cycles (according as the inverse-closed subset defining the Cayley graph consists of two involutions or a 5-cycle and its inverse). We call this graph the *strange graph* associated with $\{1,2\}$. For any $x,y,z \in \{1,\ldots,12\}$, the 3-homogeneity guarantees that the number of pairs $\{i,j\}$ for which $\{x,y\}$ and $\{y,z\}$ are edges in the associated strange graph is constant; counting shows that this constant is 1.

We seek a 10-clique among the neighbours of $\{1,2\}$. The two good 10-cliques are

$$\{\{1,i\} : i > 2\} \quad \text{and} \quad \{\{2,i\} : i > 2\}.$$

The vertices corresponding to edges of the strange graph do not form a clique; for, if $\{i,j\}$ is one, there would be a 2-arc in the strange graphs of both $\{1,2\}$ and $\{i,j\}$, contradicting the last paragraph.

So such a 10 clique must contain a pair meeting $\{1,2\}$, without loss $\{1,3\}$. The common neighbours of these two in P are $\{1,i\}$ for $i > 3$; $\{2,3\}$; two pairs of the form $\{3,x\}$ (edges in the strange graph associated with $\{1,2\}$); two of the form $\{2,x\}$; and one disjoint from $\{1,2,3\}$ (in whose strange graph $\{1,2\}$ and $\{1,3\}$ are edges). So a 9-clique in the neighbourhood of $\{1,2\}$ and $\{1,3\}$ must contain at least three pairs $\{1,i\}$ for $i > 3$.

But, if $\{x,y\}$ is a pair in such a clique, the pairs disjoint from $\{x,y\}$ are edges of a graph with valency at most 2. So every pair contains 1, and the clique is good, as claimed.

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